Exercise 27

Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.

Solution

Equation 5 is on page 262.

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

Part (a)

Let

$$y = \tanh^{-1} x.$$

Then

$$\begin{aligned} x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{\frac{e^y - e^{-y}}{2}}{\frac{e^y + e^{-y}}{2}} &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\ &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y} \\ &= \frac{e^{2y} - 1}{e^{2y} + 1} \end{aligned}$$

Solve for y by multiplying both sides by $e^{2y} + 1$.

$$\begin{aligned} x(e^{2y} + 1) &= e^{2y} - 1 \\ xe^{2y} + x &= e^{2y} - 1 \\ x + 1 &= e^{2y} - xe^{2y} \\ 1 + x &= (1 - x)e^{2y} \\ \frac{1+x}{1-x} &= e^{2y} \end{aligned}$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln \left(\frac{1+x}{1-x} \right) &= \ln e^{2y} \\ &= 2y \ln e \\ &= 2y \end{aligned}$$

Divide both sides by 2.

$$y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

Therefore,

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).$$

Part (b)

Let

$$y = \tanh^{-1} x.$$

Then

$$\tanh y = x.$$

From Exercise 18,

$$\frac{1 + \tanh y}{1 - \tanh y} = e^{2y}.$$

As a result,

$$\frac{1 + x}{1 - x} = e^{2y}.$$

Take the natural logarithm of both sides.

$$\begin{aligned} \ln\left(\frac{1 + x}{1 - x}\right) &= \ln e^{2y} \\ &= 2y \ln e \\ &= 2y \end{aligned}$$

Divide both sides by 2.

$$y = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

Therefore,

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right).$$